

$\mathcal{N} = 1$ Euler Anomaly from RG-dependent metric-Background

Vladimir Prochazka¹ and Roman Zwicky^{1,a}

¹*Higgs Centre for Theoretical Physics, School of Physics and Astronomy,
University of Edinburgh, Edinburgh EH9 3JZ, Scotland, UK*

Abstract. We consider $\mathcal{N} = 1$ supersymmetric gauge theories in the conformal window. By applying a suitable matter superfield rescaling and a Weyl-transformation the renormalisation group running (matter and gauge field Z -factors) are absorbed into the metric. The latter becomes a function of the Z -factors. The Euler flow $\Delta a \equiv a_{UV} - a_{IR}|_{\mathcal{N}=1}$ is then obtained by free field theory computation with the non-trivial dynamics coming from expanding the Euler invariant in the flow dependent metric. The result is therefore directly obtained in terms of the infrared anomalous dimension confirming an earlier result using the matching of conserved currents.

1 Introduction

For a theory on a curved space in 4D, with no explicit scale symmetry breaking, the trace of the energy momentum tensor TEMT is parametrised by [1, 2]

$$\langle \Theta \rangle = a E_4 + b W^2 + c H^2 + c' \square H, \quad H \equiv \frac{R}{d-1}, \quad (1)$$

where E_4 , W^2 and R are the topological Euler term, the Weyl tensor squared and the Ricci scalar respectively. The Euler anomaly has a long history in relation to the a-theorem, e.g. [3–5] and [6] for a review. The theorem states that the difference of the ultraviolet (UV) and infrared (IR) Euler anomaly $\Delta a \equiv a_{UV} - a_{IR}$ is strictly positive and intuitively measures the loss of degrees of freedom during the flow. For $\mathcal{N} = 1$ supersymmetric gauge theories an exact expression was derived, valid in the conformal window, [7] almost twenty years ago. Ever since it has served as a fruitful laboratory for testing different techniques by rederiving the result. Examples include four-loop perturbation theory [8], the local renormalisation group (RG) [9] and employing superspace techniques assuming a gradient flow equation [10] conforming an older conjecture [11].

Here we derive $\Delta a|_{\mathcal{N}=1}$ by absorbing the matter and gauge Z -factors into the background metric. This renders the theory, in the vacuum sector, equivalent to a free field theory in a curved background carrying the information of the dynamics $\tilde{g}_{\rho\lambda} = f(Z, \gamma_*) \delta_{\rho\lambda}$. The difference of the Euler anomaly is computed using the conformal anomaly matching and dilaton effective action techniques used by Komargodski and Schwimmer (KS) [5, 12] to prove the a-theorem but at the same time differs substantially from it.

^ae-mail: roman.zwicky@ed.ac.uk – Preprint number: CP3-Origins-2016-053 DNR90

2 Framework

In this section we derive the formula for computing Δa . Let us consider a massless theory with a generic fields ϕ and coupling g . The IR effective W is given by

$$e^{W(g(\mu), \mu)} = \int [\mathcal{D}\phi]_\mu e^{-S_W(g(\mu), \mu, \phi)}, \quad (2)$$

where the subscript W on the action stands for Wilsonian. Hence S_W is on a renormalisation trajectory from the UV to the IR fixed point. A generic Weyl rescaling of the metric $g_{\mu\nu} \rightarrow e^{-2\alpha(x)} g_{\mu\nu}$ results in $q^2 \rightarrow q^2 e^{2\alpha(x)}$ and since the correlation functions depend on ratios of q^2/μ^2 the operation is equivalent to $\mu \rightarrow e^{-\alpha(x)} \mu$. The key idea in [5] is to compensate for this change by coupling a dilaton to the RG scale $\mu e^{\tau(x)}$ which transforms as $\tau(x) \rightarrow \tau(x) + \alpha(x)$. The action in (2) then assumes the form $S_W(g(\mu), \mu, \phi) \rightarrow S_W^{(\tau)}(g(\mu e^\tau), \mu e^\tau, \phi)$ and is formally dilation invariant. The partition function assumes the form

$$e^{W_\tau(g(\mu), \mu, \tau(x))} = \int [\mathcal{D}\phi]_\mu e^{-S_W^{(\tau)}(g(\mu e^\tau), \mu e^\tau, \phi)}, \quad (3)$$

and becomes a function of the *external* dilaton field. The dilaton is the spurion of the RG transformation and its response (effective action) must be sensitive to the difference in UV and IR conformal anomalies. It was shown in [5] that the relevant term is

$$W_\tau(g_{UV}^*) - W_\tau(g_{IR}^*) = \int_{g_{IR}^*}^{g_{UV}^*} dg \partial_g W_\tau = - \int_{-\infty}^{\infty} d \ln \mu \partial_{\ln \mu} W_\tau = -\Delta a S_{WZ} + \dots, \quad (4)$$

where the Wess-Zumino action is given by

$$S_{WZ} = \int d^4x \, 2 \left(2\Box\tau(\partial\tau)^2 - (\partial\tau)^4 \right) + O(R). \quad (5)$$

and since $\partial_{\ln \mu} W_\tau = \int d^4x \sqrt{\tilde{g}} \langle \Theta \rangle_\tau$ (e.g. appendix [13]) one arrives at the key formula

$$\Delta a = \int_{-\infty}^{\infty} d \ln \mu \int d^4x \sqrt{\tilde{g}} \langle \Theta \rangle_\tau |_{S_{WZ}}, \quad (6)$$

where $|_{S_{WZ}}$ denotes projection and $\langle \Theta \rangle_\tau$ is the TEMT in the dilaton background. Assuming a conformally flat metric $\tilde{g}_{\rho\lambda} = e^{-2s(\tau)} \delta_{\rho\lambda}$ the latter reads

$$\langle \Theta \rangle_\tau = \tilde{a} \tilde{E}_4 + \tilde{c} \tilde{H}^2 + \tilde{c}' \Box H = \langle \Theta \rangle_\tau |_{S_{WZ}} S_{WZ} + \dots, \quad (7)$$

where it was used that the Weyl tensor vanishes in those spaces since it is sensitive to the spin 2 part only and we discard the $\Box \tilde{H}$ -term since being a total derivative it does not contribute S_{WZ} . Hence determining Δa reduces to finding \tilde{a} and \tilde{c} which of course depend on the content of the theory. Before turning to the $\mathcal{N} = 1$ -computation it is instructive and useful to first discuss the case of a single scalar field.

2.1 The scalar field with dynamics absorbed in the metric as an example

Consider a scalar field theory in flat space with Wilsonian action

$$S_W(\mu) = \int d^4x Z(\mu) \delta^{\rho\lambda} \partial_\rho \phi \partial_\lambda \phi + \dots, \quad (8)$$

where it is going to be sufficient, for later purposes, to focus on the kinetic term only. The dilaton is implemented by $Z(\mu) \rightarrow Z(\mu e^{\tau(x)})$ and it is the key idea of this paper that the entire RG running $Z(\mu e^{\tau(x)})$ can be absorbed into metric by the Weyl rescaling with $\alpha(x) = s(x)$ given by

$$s(\mu e^{\tau(x)}) = -\frac{1}{2} \ln Z(\mu e^{\tau(x)}) \Rightarrow \tilde{g}_{\rho\lambda} = e^{-2s(x)} \delta_{\rho\lambda} = Z \delta_{\rho\lambda}. \quad (9)$$

The theory then becomes a free field theory $S_W^{(\tau)}(\mu) = \int d^4x \sqrt{\tilde{g}} \tilde{g}^{\rho\lambda} D_\rho^{(s)} \phi D_\lambda^{(s)} \phi$ on a conformally flat space with metric $\tilde{g}_{\mu\nu}$ (9). In the previous formula $D_\rho^{(s)} = \partial_\rho - (\partial_\rho s)$ are covariant derivatives ensuring local Weyl invariance, a technical detail for which we refer the reader to [13].

To get the Euler anomaly we then need to determine (7) in a free field theory and expand the corresponding geometric term which will involve the dynamics in terms of the anomalous dimension (appearing through derivatives of the absorbed Z-factor). The TEMT in a free theory (cf. [1, 14] or the explicit computation in appendix [13]) is given by

$$\langle \Theta \rangle_\tau = a_{(0)}^{\text{free}} (\tilde{E}_4 - 2\tilde{\square} \tilde{R}), \quad a_{(0)}^{\text{free}} = \frac{1}{360} \frac{1}{16\pi^2} = \frac{1}{5760\pi^2}, \quad (10)$$

where the absence of the R^2 -term is a consequence of the free field theory being a conformal field theory. As previously explained the $\square R$ -term does not contribute to S_{WZ} and can therefore be discarded. Evaluating the latter in the background (9) leads to

$$\begin{aligned} \sqrt{\tilde{g}} \tilde{E}_4 &= -8 \left(\frac{1}{2} \square (\partial s)^2 - \partial \cdot (\partial s (\square s - (\partial s)^2)) \right) \\ &= -[\gamma^2 \square (\partial \tau)^2 + (2\gamma \dot{\gamma} - 2\gamma^2) \partial^\lambda (\partial_\lambda \tau \square \tau) - \gamma^3 \partial^\lambda (\partial_\lambda \tau (\partial \tau)^2) - 6\gamma \dot{\gamma} (\partial \tau)^2 \square \tau - 3\gamma^2 \dot{\gamma} (\partial \tau)^4], \end{aligned} \quad (11)$$

where the following abbreviation $\dot{\gamma} \equiv \frac{d}{d \log \mu} \gamma$ was used as well as

$$\partial_\rho \gamma = \dot{\gamma} \partial_\rho \tau, \quad \partial_\rho s = -\frac{1}{2} \frac{\partial \ln Z(\mu e^\tau)}{\partial (\mu e^\tau)} \partial_\rho (\mu e^\tau) = -\frac{1}{2} \gamma \partial_\rho \tau, \quad \gamma = \frac{\partial \ln Z(\mu)}{\partial \ln \mu}. \quad (12)$$

Note that γ and $\dot{\gamma}$ can be treated as being space-independent, since expanding $\gamma(\mu e^\tau) = \gamma(\mu) + O(\tau(x))$ leads to terms which are not in S_{WZ} (5). Furthermore it is then clear that the total derivative terms in (11) can be discarded since they do not contribute to the bulk-term S_{WZ} (5). In order to project on S_{WZ} it is convenient (following [5],[12]) to set $\square \tau = (\partial \tau)^2$ under which $S_{\text{WZ}} \rightarrow \int d^4x 2(\square \tau)^2$. Using (6) one gets the simple result

$$\Delta a = \frac{1}{2} \left((\gamma_{\text{UV}}^3 - \gamma_{\text{IR}}^3) + 3(\gamma_{\text{UV}}^2 - \gamma_{\text{IR}}^2) \right) a_{(0)}^{\text{free}}, \quad (13)$$

from the last two terms in (11). Above $\gamma_{\text{IR,UV}} \equiv \gamma(g_{\text{IR,UV}}^*)$ are the values of the anomalous dimensions at the respective fixed points. Eq. (13) constitutes an important intermediate result for the derivation of $\Delta a|_{N=1}$.

3 $\mathcal{N} = 1$ supersymmetric gauge theory

We aim to compute Δa for an $\mathcal{N} = 1$ supersymmetric gauge theory with $SU(N_f) \times SU(N_f)$ flavour symmetry and gauge group $SU(N_c)$. In the superfield formalism the action takes the form, e.g. [15],

$$S_{\text{W}}(\mu) = \int d^6z \frac{1}{g^2(\mu)} \text{tr} W^2 + \text{h.c.} + \frac{1}{8} Z(\mu) \sum_f \left[\int d^8z \Phi_f^\dagger e^{-2V} \Phi_f + \{\tilde{\Phi}_f \leftrightarrow \Phi_f\} \right], \quad (14)$$

with vector V and matter $(\Phi_f, \tilde{\Phi}_f)$ superfields. Above W^2 is the supersymmetric gauge field kinetic term (function of V), g is the holomorphic coupling constant and d^6z and d^8z include integration over the fermionic superspace variables. The main tool in deriving $\Delta a|_{N=1}$ is the use of the Konishi anomaly [16–18] which is also of use in deriving the NSVZ β -function e.g. [15] or the appendix of [13]. Using the Konishi anomaly, and a suitable Weyl rescaling the action (14) can be cast into a form that makes it possible to use the previously studied free field theory example.

3.1 $\Delta a|_{N=1}$ from Dilaton effective Action and Konishi anomaly

Using arguments of holomorphicity it can be argued that the running of the coupling g , of the Wilsonian effective action of the supersymmetric gauge theory (14), is one-loop exact [15, 19] and reads

$$S_W(\mu) = \frac{Z_{1/g^2}(\mu, \mu')}{g^2(\mu')} \int d^6z \text{tr} W^2 + \text{h.c.} + \frac{1}{8} Z(\mu, \mu') \sum_f \left[\int d^8z \Phi_f^\dagger e^{-2V} \Phi_f + \{\tilde{\Phi}_f \leftrightarrow \Phi_f\} \right], \quad (15)$$

where

$$Z_{1/g^2} = 1 + \gamma_* N_f \frac{g(\mu')^2}{8\pi^2} \ln \frac{\mu'}{\mu}, \quad \gamma_* N_f = -b_0 = -(3N_c - N_f), \quad (16)$$

is the Z-factor of the $1/g^2$ -coupling, γ_* the IR squark anomalous dimension and the arbitrary scale $\mu' > \mu$ can be identified with the UV cut-off Λ_{UV} . We emphasise that $S_W(\mu)$ is μ' -independent. The Konishi anomaly¹ allows, by rescaling the matter fields

$$(\Phi_f, \tilde{\Phi}_f) \rightarrow \left(\frac{\mu'}{\mu} \right)^{\gamma_*/2} (\Phi_f, \tilde{\Phi}_f), \quad (17)$$

to shift the Z_{1/g^2} -factor in front of the matter term

$$S_W(\mu) = \int d^6z \frac{1}{g(\mu')^2} \text{tr} W^2 + \text{h.c.} + \frac{1}{8} \sum_f \left[\int d^8z \hat{Z}(\mu, \mu') \Phi_f^\dagger e^{-2V} \Phi_f + \{\tilde{\Phi}_f \leftrightarrow \Phi_f\} \right], \quad (18)$$

with

$$\hat{Z}(\mu, \mu') \equiv Z(\mu, \mu') \left(\frac{\mu'}{\mu} \right)^{\gamma_*}, \quad (19)$$

where the arbitrary scale $\mu' > \mu$ can be interpreted as a UV cut-off Λ_{UV} . Crucially, the entire RG flow is absorbed into the precoefficient $\hat{Z}(\mu, \mu')$ in front of the matter term. Eq. (18) is the analogue of the action (8) for the scalar field to the degree that the running of the theory is parametrised by a coefficient in front of the matter kinetic term. The external dilaton field is introduced as before

$$\hat{Z}(\mu, \mu') \rightarrow \hat{Z}(\mu e^{\tau(x)}, \mu') = Z(\mu e^{\tau(x)}, \mu') \left(\frac{\mu'}{\mu e^{\tau(x)}} \right)^{\gamma_*}. \quad (20)$$

The crucial point is that the W^2 -term is Weyl invariant where the matter term is Weyl variant which allows to absorb the dynamics into the metric, i.e. $\hat{Z}(\mu, \mu')$. This is achieved by the following Weyl rescaling $\alpha = s^2$

$$\tilde{g}_{\rho\lambda} = \tilde{e}_\rho^a \tilde{e}_\lambda^a = e^{-2s(\mu e^{\tau(x)})} \delta_{\rho\lambda}, \quad s(\mu e^{\tau(x)}) = -\frac{1}{2} \ln \hat{Z}(\mu e^{\tau(x)}, \mu'), \quad (21)$$

¹The Konishi anomaly accounts for the rescaling of individual field, in a regularisation with a cut-off, and therefore contributes to the analogue of the field strength term (i.e. W^2) as dictated by the trace anomaly. It is not peculiar to supersymmetry but the power within supersymmetry is that it is one-loop exact since it is in the same multiplet as the chiral anomaly and therefore inherits its topological protection.

²The procedure can be kept manifestly supersymmetric, following [20], by promoting the dilaton to a (chiral) superfield T and a superfield A playing the role of the supersymmetric Weyl parameter. We refer to [13] for details.

where \tilde{e}_ρ^a is the vielbein and $\tilde{g}_{\rho\lambda}$ is the background carrying the dynamics and the remaining part is a free field theory.³

Since the UV scale μ' is arbitrary the physical quantity $\langle\Theta\rangle_\tau$ is independent of it and we may therefore send it to infinity. Since the geometric terms \tilde{E}_4 and \tilde{H}^2 are independent of μ' ,⁴ the form of (7) implies that \tilde{a} and \tilde{c} are μ' -independent and therefore constants. This means that \tilde{a} and \tilde{c} take on the values at the (free) UV fixed point and the geometric quantities are to be evaluated using the background metric $\tilde{g}_{\rho\lambda}$. This allows us to largely reuse the computation in section 2.1 as outlined below.

The steps in completing the computation are as follows. Equivalence to the example in the previous section is attained by replacing $Z \rightarrow \hat{Z}$ in Eqs. (9) to (13), (following from (21)) implying $\gamma \rightarrow \delta\gamma \equiv \gamma - \gamma_*$ and accounting for the correct number of degrees of freedom ν . Note that only matter fields contribute to the latter since \hat{Z} only stands in front of the matter term. The matter superfield consists of a complex scalar and a Weyl fermion which contribute [1]

$$\nu \equiv 2 \Big|_{\text{C-scalar}} + \frac{11}{2} \Big|_{\text{Weyl-fermion}} = \frac{15}{2} \quad (22)$$

in units of a real scalar field. This number has to be multiplied by by flavour and colour numbers: $2N_f$ (two matter-field per flavour) and N_c (the $SU(N_c)$ Casimir of the adjoint representation). Finally Δa is given by $2N_f N_c \nu \Delta a|_{\gamma_{UV,IR} \rightarrow \delta\gamma_{UV,IR}}^{(13)}$. With $(\gamma_{UV}, \gamma_{IR}) = (0, \gamma_*)$ leads to $(\delta\gamma_{UV}, \delta\gamma_{IR}) = (-\gamma_*, 0)$ and finally

$$\Delta a|_{N=1} = \frac{15}{2} N_c N_f (-\gamma_*^3 + 3\gamma_*^2) a_{(0)}^{\text{free}}. \quad (23)$$

The result (23) is indeed the same as the non-perturbative result quoted in (Eq.4.18) in [7] when taking into account the explicit form of γ_* . The formula (23) is valid in the conformal window $3/2N_c < N_f < 3N_c$ where the theory is asymptotically free in the UV and the acquires a non-trivial fixed point in the IR. Within these boundaries the anomalous dimension γ_* takes on the values -1 to 0 and Δa is manifestly positive in accordance with the a-theorem. Within $N = 1$ supersymmetric theories the a-theorem has many applications including the so-called a -maximization [21].

4 Discussion

The main result of this work is the computation of $\Delta a|_{N=1}$ (23) in the new framework where the dynamics, i.e. the Z -factor and γ_* the anomalous dimension at the IR fixed point, is absorbed into the metric (21). This renders the computation equivalent to the Weyl anomaly of a free field theory with non-trivial dynamics originating from expanding the curvature term E_4 in terms of the metric (21). A possible advantage of our approach over the original derivation [7] is that the result is directly obtained in terms of the anomalous dimension of the squarks at the IR fixed point rather than identifying the latter indirectly from a result in terms of N_c and N_f . Let us conclude by saying that the procedure is reminiscent of the AdS/CFT set-up in that dynamics is encoded in the metric.

RZ would like to thank the Confinement'16 organisers and participants for a memorable event in Thessaloniki.

³Note this is only true in the vacuum sector as the introduction of correlation function in terms of sources would involve further changes under the Weyl-transformation.

⁴To see this note that these terms depend on derivatives of s only (c.f (11)). The latter are related to the anomalous dimension $\gamma(\mu e^\tau)$ through the relation (12) which is independent of μ' .

References

- [1] S. Christensen, M. Duff, Nucl.Phys. **B154**, 301 (1979)
- [2] M. Duff, Class.Quant.Grav. **11**, 1387 (1994), hep-th/9308075
- [3] J.L. Cardy, Phys. Lett. **B215**, 749 (1988)
- [4] I. Jack, H. Osborn, Nucl.Phys. **B343**, 647 (1990)
- [5] Z. Komargodski, A. Schwimmer, JHEP **1112**, 099 (2011), 1107.3987
- [6] G.M. Shore (2016), 1601.06662
- [7] D. Anselmi, D. Freedman, M.T. Grisaru, A. Johansen, Nucl.Phys. **B526**, 543 (1998), hep-th/9708042
- [8] I. Jack, C. Poole, JHEP **01**, 138 (2015), 1411.1301
- [9] F. Baume, B. Keren-Zur, R. Rattazzi, L. Vitale, JHEP **08**, 152 (2014), 1401.5983
- [10] R. Auzzi, B. Keren-Zur, JHEP **05**, 150 (2015), 1502.05962
- [11] D.Z. Freedman, H. Osborn, Phys. Lett. **B432**, 353 (1998), hep-th/9804101
- [12] Z. Komargodski, JHEP **07**, 069 (2012), 1112.4538
- [13] V. Prochazka, R. Zwicky, JHEP **01**, 041 (2016), 1511.03868
- [14] N. Birrell, P. Davies, *Quantum Fields in Curved Space*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, 1982), ISBN 9781107392816, <https://books.google.co.uk/books?id=-b4LAQAAQBAJ>
- [15] M. Shifman, *Advanced topics in quantum field theory*. (Cambridge Univ. Press, Cambridge, UK, 2012), ISBN 9781139210362, 9780521190848, <http://www.cambridge.org/mw/academic/subjects/physics/theoretical-physics-and-ma>
- [16] K. Konishi, Phys.Lett. **B135**, 439 (1984)
- [17] K.i. Konishi, K.i. Shizuya, Nuovo Cim. **A90**, 111 (1985)
- [18] G.M. Shore, G. Veneziano, Int. J. Mod. Phys. **A1**, 499 (1986)
- [19] N. Arkani-Hamed, H. Murayama, JHEP **06**, 030 (2000), hep-th/9707133
- [20] A. Schwimmer, S. Theisen, Nucl.Phys. **B847**, 590 (2011), 1011.0696
- [21] K.A. Intriligator, B. Wecht, Nucl. Phys. **B667**, 183 (2003), hep-th/0304128